Computer and Automation Institute,
Hungarian Academy of Sciences

Research Group of Operations Research
and Decision Systems

H-1518 Budapest, P.O.Box 63,
Hungary

Ranking decision making units based on DEA-like nonreciprocal pairwise comparisons

János Fülöp and Rita Markovits-Somogyi

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János Fülöp [*] Rita Markovits-Somogyi [†]

Abstract

Ever since the birth of data envelopment analysis (DEA) the question of ranking the decision making units (DMUs) has been one of the focal points of research in the area. Among several other approaches, promising attempts have been made to marry DEA with the analytic hierarchy process (AHP) method. Keeping the idea of using DEA-based pairwise comparisons between the DMUs, as proposed in some DEA-AHP variants published in the literature, a new method is presented for combining DEA with techniques for eliciting weights from pairwise comparison matrices. The basic idea is to apply a variant of the CCR problem instead of the classic one. The ensuing scores are then utilized to build a nonreciprocal pairwise comparison matrix which serves as the basis of eliciting the ranking values of the DMUs. Main advantage of this new method is the wider range of the resulting ranking values which subsequently leads to better distinction between the DMUs. Beside the eigenvector method, optimization based methods are also considered for eliciting the ranking values from the nonreciprocal pairwise comparison matrix. Numerical examples are supplied for comparing the proposed techniques.

Keywords: Data envelopment analysis, ranking decision making units, pairwise comparisons, techniques for eliciting weights

[*]Research Group of Operations Research and Decision Systems, Computer and Automation Research Institute, Hungarian Academy of Sciences, Kende u. 13-17., H-1111 Budapest, Hungary, E-mail: fulop@sztaki.hu

[†]Department of Transport Economics, Budapest University of Technology and Economics, Bertalan L. u. 2., H-1111 Budapest, Hungary, E-mail: rsomogyi@kgazd.bme.hu

1 Introduction

Data envelopment analysis (DEA) is a very powerful tool for the efficiency evaluation of decision making units (DMUs) with multiple inputs and outputs. One of the main advantages of this non-parametric linear programming method is its capability of discerning the efficient DMUs by creating an efficiency frontier as based on the observed data and thus not requiring any *a priori* information about the relationship between the inputs and outputs.

One of its shortcomings, however, is its inability to fully rank the decision making units. Ever since it has been created by Charnes, Cooper and Rhodes [9] on the basis of the idea of Farrel [18], the question of full ranking has been in the frontline of research.

A multitude of different DEA variants with diverse backgrounds have been developed with the aim of solving this problem. Perhaps the most widely known and applied ranking method is the super-efficiency DEA model. Developed by Andersen and Petersen [3], this technique creates the best practice frontier first without evaluating DMU_0 and then with its inclusion. Next the extent to which the envelopment frontier becomes extended is investigated. Several extensions and variants of this method are available, e.g. [24, 29, 28]. The problem with super-efficiency DEA is that under certain conditions infeasibility occurs which limits the applicability of the technique. (For details see [35]).

Another approach is that of cross-efficiency introduced by Sexton et al. [37] and extended by Doyle and Green [15] where the individual decision making units are not only assessed by their own weights but the weights of all the other DMUs are also incorporated into the value judgement. This score is more representative of efficiency than the traditional DEA-score but at the same time the connection to the multiplier weights is lost [2].

Torgensen et al. [40] investigate the extent to which the different DMUs are peers to each other and through this benchmarking procedure is full ranking achieved.

If developed purposefully, the utilization of common weights can contribute to reaching a full ranking as well. The main goal of Wang et al. [41] is to introduce a minimum weight restriction and as a side effect, common weights and then, full ranking is also achieved. Common weights are also achieved by different multivariate statistical analyses and these can also lead to full ranking. For instance, canonical correlation, linear discriminant analysis or the discriminant analysis of ratios can also be employed [2].

Another way of approaching the question of full ranking is taking into account the slacks present in the slack-adjusted DEA model. Bardhan et al. [4] rank inefficient units this way, Tone [39] composes a method that can rank all the DMUs. Du et al. [16] create an extension to this model and Chen and Sherman [10] also use slacks for the development of a non-radial super-efficiency DEA model.

Wen and Li [42] aim to utilize fuzzy information in data envelopment analysis and as a side effect full ranking is achieved.

Multi-criteria decision making methods can also be combined with DEA to provide full ranking. Sinuany-Stern et al. [38] integrate analytic hierarchy process with DEA: the pairwise comparison matrix of AHP is created through the objective evaluation of pairs of DMUs by DEA.

The method developed by Sinuany-Stern et al. [38] has been applied for instance by Guo et al. [21] for supply chain evaluation. Royendegh and Erol [33] also build upon the idea of [38] but extend the method to analytic network process (ANP), the more generalized form of AHP. Zhang et al. [44] combine DEA with AHP for 4PL vendor selection but their approach is different. After the construction of an input-output structure AHP is utilized for a preliminary data analysis with the help of which the importance of the different criteria is determined. The results of the AHP are then used as preferential information in a modified DEA model. A pairwise comparison matrix is created with the evolving efficiency values and then AHP is applied again for the evaluation of the matrix.

The authors have found the work of Sinuany-Stern et al. [38] particularly inspiring and appreciating the results therein, wish to further improve the original method by enabling the pairwise comparison matrix to be nonreciprocal which contributes to the possibility of a more accurate evaluation. Even more so, since in the course of applying DEA to the case of logistic centres [30], it has been revealed that the thumb rule in connection with the number of DMUs to be found in the literature is very difficult to be adhered to. According to this rule the number of observations should be three times greater than the number of the inputs plus outputs; and the number of DMUs should be equal or larger than the product of the number of inputs and outputs [5]. Some authors are less strict in their conduct, Wu and Goh [43] for example argue that the number of DMUs should only be minimally two times as much as the sum of the number of inputs and outputs. However, under certain conditions even this requirement might be difficult to satisfy.

The new technique proposed in this article intends to provide a solution for both the problem of full ranking and the difficulty inherent in the thumb rule cited above. Nonetheless, it is not in the scope of the present paper to explore and compare all the DEA-connected techniques aimed at resolving the question of full ranking, or even to measure up the proposed method to all the rest of existing solutions. It is its goal, however, to provide numerical examples of its utilization, with a special attention to the cases presented in articles closely related to the technique at hand.

In Section 2, a variant of the CCR problem is proposed instead of the classic one used in [38]. Easy ways for computing the related reciprocal and nonreciprocal pairwise comparison matrices are also presented. Section 3 addresses the issue how to elicit ranking weights from the nonreciprocal pairwise comparison matrix obtained by the new approach. Beside the eigenvector method used in [38], the logarithmic least squares method and the weighted least squares method are also considered. In Section 4, numerical examples are supplied for comparing the proposed techniques.

2 DEA-like pairwise comparisons of the decision making units

Let us assume that there are n decision making units each producing s different outputs from m different inputs. X_{ij} is the input i of unit j, while Y_{rj} is the output r of unit j. We assume that all X_{ij}s and Y_{rj}s are positive. In the original approach of Sinuany-Stern et al. [38] traditional DEA runs are executed for any pair of DMUs, as if only these two decision making units existed. The runs are based on the DEA CCR model adapted to the case of two DMUs. For this let A and B be a pair of units and let us consider the CCR model as if only these two units existed:

$$
\begin{aligned}
F_{\mathrm{AB}} \quad = \quad \max \quad & \sum_{r=1}^{s} u_r Y_{r\mathrm{A}} \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i X_{i\mathrm{A}} = 1, \\
& \sum_{r=1}^{s} u_r Y_{r\mathrm{A}} \leq 1, \\
& \sum_{r=1}^{s} u_r Y_{r\mathrm{B}} - \sum_{i=1}^{m} v_i X_{i\mathrm{B}} \leq 0, \\
& u_r \geq 0, r = 1, \ldots, s, \;\; v_i \geq 0, i = 1, \ldots, m.
\end{aligned}
\tag{1}
$$

In [38] the notation E_{AA} is used for F_{AB}. We think however that F_{AB} is a more appropriate notation since both units appear in it, and the precedence of A over B means that it is the efficiency of A which is being evaluated by using the two units. The change from E to F is necessary in order to avoid its confusion with another efficiency value later on. Consider another problem being in a close relation to (1):

$$
\begin{array}{rcll}
\hat{F}_{\mathrm{AB}} & = & \max & \sum\limits_{r=1}^{s} u_r Y_{r\mathrm{A}} \\
 & & \text{s.t.} & \sum\limits_{i=1}^{m} v_i X_{i\mathrm{A}} = 1, \\
 & & & \sum\limits_{r=1}^{s} u_r Y_{r\mathrm{B}} - \sum\limits_{i=1}^{m} v_i X_{i\mathrm{B}} = 0, \\
 & & & u_r \geq 0, r = 1, \ldots, s, \;\; v_i \geq 0, i = 1, \ldots, m.
\end{array} \tag{2}
$$

Comparing (2) with (1), the main difference between the two models becomes clearly visible: the second constraint of (1) representing an upper bound for the objective function of (1) is omitted. The reason behind this is the basic idea of the new model (2): the aim with the exclusion is to provide an opportunity for a full comparison between the two decision making units, without limiting the evolving score. If that constraint is left untouched, the resulting efficiency value will very frequently be the unity; and thus real distinction is not achieved between the two DMUs. A similar idea of omitting the upper bound on the objective function of (1) has already appeared in case of the super-efficiency ranking techniques [2, 3] too.

A further minor remark concerns the inequality in the third constraint of (1) which changes to equality in (2). This can be explained by the following: should we leave the inequality in (2), it would clearly hold as an equality for any optimal solution of (2). Since we shall only be interested in the optimal solutions of (2), we can consider that constraint as an equality.

Proposition 1:

$$
\hat{F}_{\mathrm{AB}} = \max_{(r,i)} \frac{Y_{r\mathrm{A}}/X_{i\mathrm{A}}}{Y_{r\mathrm{B}}/X_{i\mathrm{B}}} = \max_{r=1,\ldots,s} \frac{Y_{r\mathrm{A}}}{Y_{r\mathrm{B}}} \cdot \max_{i=1,\ldots,m} \frac{X_{i\mathrm{B}}}{X_{i\mathrm{A}}} \tag{3}
$$

and

$$
F_{\mathrm{AB}} = \min\{1, \hat{F}_{\mathrm{AB}}\}. \tag{4}
$$

Proof: Problem (2) has a finite optimal solution taken at a basic feasible solution. It is clear from the special structure of (2) that any basic feasible solution has exactly two positive variables: one from the v_i and one from the u_r variables. Given a basic feasible solution of (2), let i_0 and r_0 denote the indices of those positive variables. It is easy to see that

$$
v_{i_0} = 1/X_{i_0\mathrm{A}} \quad \text{and} \quad u_{r_0} = v_{i_0} X_{i_0\mathrm{B}}/Y_{r_0\mathrm{B}} = \frac{X_{i_0\mathrm{B}}/X_{i_0\mathrm{A}}}{Y_{r_0\mathrm{B}}},
$$

and the value of the objective function is

$$\frac{Y_{r_0\mathrm{A}}/X_{i_0\mathrm{A}}}{Y_{r_0\mathrm{B}}/X_{i_0\mathrm{B}}}. \tag{5}$$

Finding the optimal basic solution means finding the pair (r_0, i_0) with the maximal value of (5). This implies (3) directly.

Keeping in mind the upper bounding role of the second constraint of (1), it is evident that F_{AB} is equal to $\hat{F}_{\mathrm{AB}}$ if $\hat{F}_{\mathrm{AB}} \leq 1$, and $F_{\mathrm{AB}} = 1$ otherwise. □

In essence, $\hat{F}_{\mathrm{AB}}$ is the resulting value of the pairwise comparison, it represents the efficiency of A in comparison with B. According to (3), it can also be interpreted as the product of an output and an input efficiency value. The first is the maximal output ratio of A compared with B, and the second one is the maximal input ratio of B compared with A. This can also be interpreted in a way that in this comparison that single input and that single output will be chosen which is most satisfying from unit A's point of view. It is clear from (3) that this pairwise comparison is not reciprocal, i.e. $\hat{F}_{\mathrm{AB}} = 1/\hat{F}_{\mathrm{BA}}$ does not necessarily hold.

The original approach proposed in [38] determines F_{AB} in the first step, then, in the second step a cross evaluation of unit B is performed based on the idea of [32]:

$$\begin{aligned} E_{\mathrm{BA}} \;=\; \max \quad & \sum_{r=1}^{s} u_r Y_{r\mathrm{B}} \\ \text{s.t.} \quad & \sum_{i=1}^{m} v_i X_{i\mathrm{B}} = 1, \\ & \sum_{r=1}^{s} u_r Y_{r\mathrm{B}} \leq 1, \\ & \sum_{r=1}^{s} u_r Y_{r\mathrm{A}} - F_{\mathrm{AB}} \sum_{i=1}^{m} v_i X_{i\mathrm{A}} = 0, \\ & u_r \geq 0, r = 1, \ldots, s, \;\; v_i \geq 0, i = 1, \ldots, m. \end{aligned} \tag{6}$$

Actually, E_{BA} is the optimal cross evaluation of unit B while the output/input ratio of unit A is fixed at F_{AB}. Using a similar reasoning as with (1) and (2), and omitting the second constraints of (2), i.e. an upper bound restriction for the objective function, the following auxiliary problem can be established for (6):

$$
\begin{array}{rcll}
\hat{E}_{\mathrm{BA}} & = & \max & \sum_{r=1}^{s} u_r Y_{r\mathrm{B}} \\
 & & \text{s.t.} & \sum_{i=1}^{m} v_i X_{i\mathrm{B}} = 1, \\
 & & & \sum_{r=1}^{s} u_r Y_{r\mathrm{A}} - F_{\mathrm{AB}} \sum_{i=1}^{m} v_i X_{i\mathrm{A}} = 0, \\
 & & & u_r \geq 0, r = 1, \ldots, s,\ \ v_i \geq 0, i = 1, \ldots, m.
\end{array} \tag{7}
$$

The following Proposition 2 can be proved in the same way as Proposition 1, only the constant F_{AB} appearing in the second constraint of (7) requires some extra attention. We leave the proof to the reader.

Proposition 2:

$$
\hat{E}_{\mathrm{BA}} = F_{\mathrm{AB}} \cdot \max_{(r,i)} \frac{Y_{r\mathrm{B}}/X_{i\mathrm{B}}}{Y_{r\mathrm{A}}/X_{i\mathrm{A}}} = F_{\mathrm{AB}} \cdot \max_{r=1,\ldots,s} \frac{Y_{r\mathrm{B}}}{Y_{r\mathrm{A}}} \cdot \max_{i=1,\ldots,m} \frac{X_{i\mathrm{A}}}{X_{i\mathrm{B}}} \tag{8}
$$

and

$$
E_{\mathrm{BA}} = \min\{1, \hat{E}_{\mathrm{BA}}\}. \tag{9}
$$

□

Collating (8) and (3), changing the role of A and B in the latter, we obtain

Corollary 1:

$$
\hat{E}_{\mathrm{BA}} = F_{\mathrm{AB}} \hat{F}_{\mathrm{BA}}. \tag{10}
$$

□

Proposition 3: If $F_{\mathrm{AB}} < 1$, then $E_{\mathrm{BA}} = 1$.

Proof: From (4) and $F_{\mathrm{AB}} < 1$, we get $F_{\mathrm{AB}} = \hat{F}_{\mathrm{AB}}$. Then, from (3) and (8),

$$
\hat{E}_{\mathrm{BA}} = \max_{(r,i)} \frac{Y_{r\mathrm{A}}/X_{i\mathrm{A}}}{Y_{r\mathrm{B}}/X_{i\mathrm{B}}} \cdot \max_{(r,i)} \frac{Y_{r\mathrm{B}}/X_{i\mathrm{B}}}{Y_{r\mathrm{A}}/X_{i\mathrm{A}}} = \frac{\max_{(r,i)} \frac{Y_{r\mathrm{A}}/X_{i\mathrm{A}}}{Y_{r\mathrm{B}}/X_{i\mathrm{B}}}}{\min_{(r,i)} \frac{Y_{r\mathrm{A}}/X_{i\mathrm{A}}}{Y_{r\mathrm{B}}/X_{i\mathrm{B}}}} \geq 1.
$$

From (9), we get $E_{\mathrm{BA}} = 1$ immediately. □

As a consequence of Proposition 3, if $F_{\mathrm{AB}} < 1$, then it is unnecessary to solve problems (6) or (7), we get $E_{\mathrm{BA}} = 1$ directly.

Proposition 4: If there exist (r_1, i_1) and (r_2, i_2) such that

$$Y_{r_1\mathrm{A}}/X_{i_1\mathrm{A}} \geq Y_{r_1\mathrm{B}}/X_{i_1\mathrm{B}} \quad \text{and} \quad Y_{r_2\mathrm{A}}/X_{i_2\mathrm{A}} \leq Y_{r_2\mathrm{B}}/X_{i_2\mathrm{B}},$$

then $F_{\mathrm{AB}} = E_{\mathrm{BA}} = 1$.
Proof: From (3) and (4), we obtain $F_{\mathrm{AB}} = 1$. Then, (8) implies $\hat{E}_{\mathrm{BA}} \geq 1$ thus $E_{\mathrm{BA}} = 1$. □

Remark 1: It is easy to see that the condition of Proposition 4 does not hold if and only if

$$Y_{r\mathrm{A}}/X_{i\mathrm{A}} < Y_{r\mathrm{B}}/X_{i\mathrm{B}} \quad \text{for all } (r, i) \tag{11}$$

or

$$Y_{r\mathrm{A}}/X_{i\mathrm{A}} > Y_{r\mathrm{B}}/X_{i\mathrm{B}} \quad \text{for all } (r, i). \tag{12}$$

Remark 2: We have also pointed out in Propositions 1 and 2 that we do not have to use any optimization software to obtain the optimal solution of (1), (2), (6) and (7).

Since n DMUs are given, and the values F_{AB}, $\hat{F}_{\mathrm{AB}}$, E_{BA} and $\hat{E}_{\mathrm{BA}}$ are to be determined for all pairs A and B of the DMUs, we introduce the following notations. Let $\mathrm{DMU}_1,\ldots, \mathrm{DMU}_n$ denote the decison making units. Considering DMU_j as A and DMU_k as B, let $F_{jk} = F_{\mathrm{AB}}$, $\hat{F}_{jk} = \hat{F}_{\mathrm{AB}}$, $E_{kj} = E_{\mathrm{BA}}$ and $\hat{E}_{kj} = \hat{E}_{\mathrm{BA}}$.

Sinuany-Stern et al. [38] construct an $n \times n$ matrix $A = [a_{jk}]$ of the entries

$$a_{jk} = \frac{F_{jk} + E_{jk}}{F_{kj} + E_{kj}}, \quad j, k = 1, \ldots, n. \tag{13}$$

The nominator of a_{jk} is the sum of the efficiency evaluation and the cross evaluation of DMU_j in comparison with DMU_k. The denominator can be interpreted similarly by changing the role of j and k.

Clearly,

$$a_{jk} > 0, \quad j, k = 1, \ldots, n, \tag{14}$$

and

$$a_{jk} = 1/a_{kj}, \quad j, k = 1, \ldots, n. \tag{15}$$

An $n \times n$ matrix A with the properties (14) and (15) is called a *pairwise comparison matrix* [34].

Constructing the pairwise comparison matrix A by (13) is, however, in some cases problematic. Namely, if considering A as DMU_j and B as DMU_k

and neither (11) nor (12) hold, then $F_{jk} = E_{kj} = F_{kj} = E_{jk} = 1$, consequently, $a_{jk} = a_{kj} = 1$. In some practical or randomly generated cases, it is not very probable that the dominance of (11) or (12) holds. This may lead to the phenomenon observed in some numerical examples applying the approach of [38] that the pairwise comparison matrices constructed by (13) comprise strikingly many 1 elements [21, 33, 38, 44]. The result 1 of a pairwise comparison means that the two DMUs cannot be considered as different. Therefore, a large number of unities in the pairwise comparison matrix may hinder the strict ranking of DMUs as the ranking weights elicited from the pairwise comparison matrix may be identical or very close to each other.

In this paper, instead of using (13), we propose to construct an $n \times n$ matrix $A = [a_{jk}]$ of the entries

$$a_{jk} = \hat{F}_{jk}, \quad j, k = 1, \dots, n. \tag{16}$$

As mentioned earlier, $\hat{F}_{jk}$ provides the opportunity to compare DMU_j against DMU_k without limiting the score from above. It is undeniable that DMU_j is in a privileged position in the course of this comparison. It can also be considered as a football match where DMU_j has the home team's advantage. Of course, the role of DMU_j and DMU_k is reversed when the value $a_{kj} = \hat{F}_{kj}$ is determined, and thus a level playing field can be assured.

It was already pointed out that $\hat{F}_{\text{AB}}$ and $1/\hat{F}_{\text{BA}}$ may be different, i.e. the reciprocity property (15) does not necessarily hold for a matrix A constructed by (16). Actually, matrix A of (16) is simply a positive matrix but we call it *nonreciprocal* pairwise comparison matrix to indicate the context.

3 Eliciting weights from the nonreciprocal pairwise comparison matrix

After having constructed the pairwise comparison matrix A by (13), Sinuany-Stern et al. [38] follow the standard AHP methodology [34]. By applying the Eigenvector Method (EM), the maximal eigenvalue $\lambda_{\max}$ and its corresponding eigenvector w^{EM} of

$$Aw = \lambda w \tag{17}$$

are determined. The real number $\lambda_{\max}$ and the vector w^{EM} are positive and unique. In addition, $\lambda_{\max} \geq n$, and $\lambda_{\max} = n$ if and only if A is consistent, i.e.

$$a_{ij}a_{jk} = a_{ik} \;\text{ for all } i, j, k = 1, \dots, n.$$

Having determined the vector w^{EM}, the DMUs are ranked in the following way. Rank 1 is assigned to the DMU with the maximal value of w_j^{EM}, and in decreasing order of w_j^{EM} are the further ranks allocated to the remaining DMUs.

In the standard AHP methodology, matrix A is assumed to be reciprocal. The matrix A of (16) is however nonreciprocal. As far as the authors know nonreciprocal matrices in a pairwise comparison context and the question of how to elicit the weight vector w from them appeared first in [27]. Although double wine testing is mentioned as a main example for the necessity of relaxation of the reciprocity condition, further examples of application from other fields of life have also been published in the literature.

The lack of reciprocity may even occur if the comparisons are performed by the same person at different times. In [14] an experiment is reported. Postgraduate students were asked to fill in the upper part of a pairwise comparison matrix, where the items to be compared were in the scope of their studies. Some weeks later, they were asked to fill in the lower part of the PC matrix. It turned out that for none of the matrices obtained in this way did the reciprocity property hold.

As mentioned in [23], nonreciprocal pairwise comparison matrices may also appear when one compares financial assets denominated in different currencies. Because of transaction costs, the resulting matrices are not reciprocal, even if no subjectivity is involved.

The classical methods used in case of reciprocal pairwise comparison matrices can also be extended to the nonreciprocal case in more or less direct ways. EM can also be interpreted without the reciprocity condition since the Perron-Frobenius and the Frobenius theorems, the mathematical bases of the EM, do not require A to be reciprocal, see [36] for details. The property $\lambda_{\max} \geq n$ does not necessarily hold for a nonreciprocal matrix A, thus, the consistency index $\text{CI} = (\lambda_{\max} - n)/(n-1)$ playing an important role in AHP [34] may be negative. However, since the pairwise comparison matrices are constructed from objective data in our case, we are not concerned about the consistency of A.

The Eigenvector Method is not the only way to elicit ranking weights from a pairwise comparison matrix. A group of approaches applies optimization methods and proposes different ways for minimizing the difference between A and consistent pairwise comparison matrices. The optimization methods are based on the basic property that A is consistent if and only if

$$a_{ij} = w_i/w_j, \quad i, j = 1, \ldots, n,$$

where w is a positive n-vector and is unique after a normalization. Most of the optimization approaches can be directly extended to the nonreciprocal case as well. If the difference to be minimized is measured in the least-squares sense, i.e., with the Frobenius norm, then we get the Least Squares Method (LSM) [11]:

$$\begin{aligned} \min \quad & \sum_{i=1}^n \sum_{j=1}^n \left(a_{ij} - \frac{w_i}{w_j}\right)^2 \\ \text{s.t.} \quad & \sum_{i=1}^n w_i = 1, \quad w_i > 0, \quad i = 1, \dots, n. \end{aligned} \tag{18}$$

Under special conditions, (18) can be transcribed into the form of a convex optimization problem and can be solved by simple local search techniques [19, 20]. However, without the special conditions, problem (18) may be a difficult nonconvex optimization problem with multiple local optima and even with multiple isolated global optimal solutions [25, 26].

In order to elude the difficulties caused by the possible nonconvexity of (18), several other, more easily solvable problem forms are proposed to derive priority weights from a pairwise comparison matrix. The Weighted Least Squares Method (WLSM) [6, 11] in the form of

$$\begin{aligned} \min \quad & \sum_{i=1}^n \sum_{j=1}^n (a_{ij} w_j - w_i)^2 \\ \text{s.t.} \quad & \sum_{i=1}^n w_i = 1, \quad w_i \geq 0, \quad i = 1, \dots, n \end{aligned} \tag{19}$$

involves a convex quadratic optimization problem whose unique optimal solution is easy to obtain.

The Logarithmic Least Squares Method (LLSM) [12, 13] in the form

$$\begin{aligned} \min \quad & \sum_{i=1}^n \sum_{i=1}^n \left[\log a_{ij} - \log\left(\frac{w_i}{w_j}\right)\right]^2 \\ \text{s.t.} \quad & \prod_{i=1}^n w_i = 1, \quad w_i > 0, \quad i = 1, \dots, n \end{aligned} \tag{20}$$

is based on an optimization problem whose unique optimal solution, in the reciprocal case, is the geometric mean of the rows of matrix A. This result was extended to the nonreciprocal case in [27], and the following optimal solution to (20) was obtained:

$$w_i = \sqrt[2n]{\frac{\prod_{j=1}^n a_{ij}}{\prod_{j=1}^n a_{ji}}}, \quad i = 1, \dots, n. \tag{21}$$

In this paper, for the numerical experiment, we used the EM, LLSM and WLSM approaches to elicit ranking weights from pairwise comparison matrices. For further approaches, see [7, 8, 17, 19] and the references therein.

Of course, the different approaches may result in different weight vectors w, and consequently, in different ranking orders of some DMUs. This well-known phenomenon occurred in the following numerical examples, too.

4 Numerical examples

We compared the original AHP/DEA ranking methodology proposed in [38] with the new method presented in this paper. The original and three variants of the new method were tested in parallel. The latter differ in the way how the weight vector w, used for ranking the DMUs, is elicited from the appropriate pairwise comparison matrix. EM1 – the original method – applies the Eigenvector Method on the reciprocal pairwise comparison matrix of elements yielded by (13), while EM2, LLSM and WLSM apply the Eigenvector Method, the Logarithmic Least Squares Method and the Weighted Least Squares Method, respectively, on the nonreciprocal pairwise comparison matrix $\hat{F}$.

Example 1.

Example 1 is the nursing home example developed in [37]. This example was also used in the review paper [2] to compare the majority of the techniques surveyed there. The example comprises six DMUs, two inputs and two outputs: staff hours per day (StHr) and supplies per day (Supp) as inputs, and total Medicare plus Medicaid reimbursed patient days (MCPD) and total private patient days (PPD) as outputs. The raw data are presented in Table 1.

Table 1 Input and output data of Example 1

DMU	Inputs		Outputs	
	StHr	Supp	MCPD	PPD
A	150	0.2	14000	3500
B	400	0.7	14000	21000
C	320	1.2	42000	10500
D	520	2.0	28000	42000
E	350	1.2	19000	25000
F	320	0.7	14000	15000

By applying (2), we obtain the matrix $\hat{F}$:

$$\hat{F} = \begin{pmatrix} 1.00000 & 3.50000 & 2.00000 & 5.00000 & 4.42105 & 3.50000 \\ 2.25000 & 1.00000 & 3.42857 & 1.42857 & 1.44000 & 1.40000 \\ 1.40625 & 3.75000 & 1.00000 & 2.50000 & 2.41776 & 3.00000 \\ 3.46154 & 1.53846 & 2.46154 & 1.00000 & 1.13077 & 1.72308 \\ 3.06122 & 1.55102 & 2.38095 & 1.13095 & 1.00000 & 1.52381 \\ 2.00893 & 1.25000 & 2.44898 & 1.42857 & 1.26316 & 1.00000 \end{pmatrix}.$$

Since every element of $\hat{F}$ is greater than or equal to 1, from (4), we get $F_{jk} = 1$, $j, k = 1, \dots, n$. Then, from (10), we obtain $\hat{E}_{kj} \geq 1$, and from (9), $E_{kj} = 1$ for all $j, k = 1, \dots, n$. Consequently, the reciprocal pairwise comparison matrix A constructed by (13) consists only of unity elements. This matrix is consistent, the maximal eigenvector consists of equal weights, and all DMUs get the ranking 1-6 as shown in Table 2 in the columns under EM1.

Methods EM2, LLSM and WLSM were run on the nonreciprocal pairwise comparison matrix $\hat{F}$. The results are also displayed in Table 2. Although a deeper numerical comparison of these methods against other ranking methods is beyond the scope of this paper, it is worth noting that the ranking by EM2 is the same as that of the super-efficiency approach in [2], moreover, the correlation of the vector of weights by EM2 to that by the super-efficiency technique is 0.99329. In the case of LLSM and WLSM, the ranking orders are the same only in the first two positions but the correlations are still 0.97891 and 0.97337, respectively.

Table 2 Weights and ranking of the DMUs in Example 1

DMU	Weights				Ranking			
	EM1	EM2	LLSM	WLSM	EM1	EM2	LLSM	WLSM
A	0.16667	0.23760	0.19755	0.19196	1-6	1	1	1
B	0.16667	0.14988	0.15839	0.15899	1-6	4	5	5
C	0.16667	0.17439	0.16565	0.16947	1-6	2	2	2
D	0.16667	0.15985	0.16437	0.14730	1-6	3	4	6
E	0.16667	0.14951	0.16438	0.16461	1-6	5	3	4
F	0.16667	0.12877	0.14966	0.16767	1-6	6	6	3

Example 2.

The raw data of Example 2 is from Table 3 of [38], and is shown in Table 3 below.

Table 3 Input and output data of Example 2

DMU	Inputs		Outputs	
	X_1	X_2	Y_1	Y_2
A	50	55	10	56
B	130	60	12	78
C	68	96	45	9
D	45	30	35	18
E	5	3	99	3

By applying (2), we obtain the matrix $\hat{F}$:

$$\hat{F} = \begin{pmatrix} 1.00000 & 2.16667 & 10.86061 & 2.80000 & 1.86667 \\ 1.27679 & 1.00000 & 13.86667 & 2.16667 & 1.30000 \\ 3.30882 & 7.16912 & 1.00000 & 0.85084 & 0.22059 \\ 6.41667 & 8.42593 & 6.40000 & 1.00000 & 0.66667 \\ 181.50000 & 214.50000 & 70.40000 & 28.28571 & 1.00000 \end{pmatrix}. \quad (22)$$

The reciprocal pairwise comparison matrix of the elements (13) is

$$A = \begin{pmatrix} 1.00000 & 1.00000 & 1.00000 & 1.00000 & 1.00000 \\ 1.00000 & 1.00000 & 1.00000 & 1.00000 & 1.00000 \\ 1.00000 & 1.00000 & 1.00000 & 0.85084 & 0.22059 \\ 1.00000 & 1.00000 & 1.17531 & 1.00000 & 0.66667 \\ 1.00000 & 1.00000 & 4.53333 & 1.50000 & 1.00000 \end{pmatrix}. \quad (23)$$

Two versions of Example 2 were solved in [38]. In the first one only four DMUs, Unit A to Unit D, were considered. The matrices $\hat{F}$ and A corresponding to this case can be easily obtained by taking the 4×4 upper-left submatrix of (22) and (23), respectively. The weights and the ranking of the DMUs in the case with four DMUs are shown in Table 4. The second version of Example 2 is with five DMUs. The corresponding weights and ranking can be found in Table 5.

Table 4 Weights and ranking of the DMUs in Example 2 with four DMUs

DMU	Weights				Ranking			
	EM1	EM2	LLSM	WLSM	EM1	EM2	LLSM	WLSM
A	0.24980	0.23732	0.26086	0.22924	2-3	3	2	2
B	0.24980	0.24654	0.20025	0.11203	2-3	2	3	3
C	0.24010	0.19087	0.14399	0.04622	4	4	4	4
D	0.26031	0.32527	0.39490	0.61251	1	1	1	1

Table 5 Weights and ranking of the DMUs in Example 2 with five DMUs

DMU	Weights				Ranking			
	EM1	EM2	LLSM	WLSM	EM1	EM2	LLSM	WLSM
A	0.19057	0.06436	0.07302	0.00538	2-3	2	3	4
B	0.19057	0.05037	0.05606	0.00451	2-3	3	4	5
C	0.14070	0.02515	0.04031	0.01349	5	5	5	3
D	0.17608	0.04872	0.11055	0.03955	4	4	2	2
E	0.30208	0.81140	0.72007	0.93708	1	1	1	1

In both versions, methods EM2, LLSM and WLSM yield full ranking orders although they are different in several positions. It is also a striking property of the methods based on the nonreciprocal pairwise comparison matrix $\hat{F}$ that the weights are more distinguished. The largest weights are significantly larger and the smallest weights are significantly smaller than those of EM1. Also, it is worth observing how the favorable input and output values of DMU E are reflected in the corresponding weights in Table 5.

Example 3.

Example 3 comes from [21] and the raw data are given in Table 6.

Table 6 Input and output data of Example 3

DMU	Inputs			Outputs		
	X_1	X_2	X_3	Y_1	Y_2	Y_3
DMU_1	15	15	0.05	0.80	0.800	0.42
DMU_2	70	25	0.10	0.90	0.900	0.53
DMU_3	45	16	0.07	0.96	0.885	0.47
DMU_4	40	30	0.12	0.85	0.750	0.32
DMU_5	35	25	0.11	0.75	0.845	0.44
DMU_6	60	18	0.15	0.85	0.755	0.25
DMU_7	55	20	0.08	0.70	0.850	0.51
DMU_8	30	12	0.09	0.95	0.700	0.46

The nonreciprocal pairwise comparison matrix $\hat{F}$ is obtained by (2), and the reciprocal A by (13):

$$\hat{F} = \begin{pmatrix} 1.00000 & 4.14815 & 2.71186 & 3.50000 & 2.48889 & 6.72000 & 4.19048 & 2.28571 \\ 0.75714 & 1.00000 & 0.78936 & 1.98750 & 1.32500 & 3.18000 & 1.02857 & 1.15714 \\ 1.12500 & 1.66667 & 1.00000 & 2.75391 & 2.01143 & 4.02857 & 1.71429 & 1.62551 \\ 0.53125 & 1.65278 & 0.99609 & 1.00000 & 1.03889 & 1.92000 & 1.66964 & 0.80357 \\ 0.63375 & 1.87778 & 1.22760 & 1.65000 & 1.00000 & 3.01714 & 1.68367 & 1.03469 \\ 0.88542 & 1.31173 & 0.78704 & 1.67778 & 1.57407 & 1.00000 & 1.34921 & 0.71905 \\ 0.91071 & 1.22470 & 0.94947 & 2.39062 & 1.59375 & 3.82500 & 1.00000 & 1.36607 \\ 1.48437 & 2.46296 & 1.48437 & 3.59375 & 2.63889 & 3.68000 & 2.48810 & 1.00000 \end{pmatrix},$$

$$A = \begin{pmatrix} 1.00000 & 1.32075 & 1.00000 & 1.88235 & 1.57791 & 1.12941 & 1.09804 & 1.00000 \\ 0.75714 & 1.00000 & 0.78936 & 1.00000 & 1.00000 & 1.00000 & 1.00000 & 1.00000 \\ 1.00000 & 1.26685 & 1.00000 & 1.00392 & 1.00000 & 1.27059 & 1.05322 & 1.00000 \\ 0.53125 & 1.00000 & 0.99609 & 1.00000 & 1.00000 & 1.00000 & 1.00000 & 0.80357 \\ 0.63375 & 1.00000 & 1.00000 & 1.00000 & 1.00000 & 1.00000 & 1.00000 & 1.00000 \\ 0.88542 & 1.00000 & 0.78704 & 1.00000 & 1.00000 & 1.00000 & 1.00000 & 0.71905 \\ 0.91071 & 1.00000 & 0.94947 & 1.00000 & 1.00000 & 1.00000 & 1.00000 & 1.00000 \\ 1.00000 & 1.00000 & 1.00000 & 1.24444 & 1.00000 & 1.39073 & 1.00000 & 1.00000 \end{pmatrix}.$$

The weights and the ranking orders obtained by the tested methods are as follows:

Table 7 Weights and ranking of the DMUs in Example 3

DMU	Weights				Ranking			
	EM1	EM2	LLSM	WLSM	EM1	EM2	LLSM	WLSM
DMU_1	0.15290	0.22575	0.21761	0.26630	1	1	1	1
DMU_2	0.11616	0.09546	0.09978	0.08828	6	6	6	6
DMU_3	0.13279	0.13641	0.14839	0.15489	3	3	3	3
DMU_4	0.11203	0.08305	0.08474	0.07254	8	8	7	7
DMU_5	0.11729	0.10275	0.10874	0.10563	5	5	5	4
DMU_6	0.11391	0.08334	0.07143	0.04668	7	7	8	8
DMU_7	0.12172	0.11261	0.10932	0.09334	4	4	4	5
DMU_8	0.13318	0.16063	0.16001	0.17233	2	2	2	2

The ranking orders by EM1, EM2 and LLSM coincide in the first six positions. WLSM renders a further rank reversal in positions 4 and 5. The greater separation in the weights by EM2, LLSM and WLSM, in comparison to those by EM1, can be observed at this example, too. We mention that the weight vector w published in [21] is w =(0.152, 0.117, 0.134, 0.112, 0.118, 0.114, 0.122, 0.133)^T yielding the ranking (1,6,2,8,5,7,4,3). The slight difference to the weights obtained by EM1 may come from a larger stopping tolerance when computing the maximal eigenvector in [21]. But even a slight difference implies a rank reversal for DMU_3 and DMU_8.

5 Conclusion

The method proposed in this paper seems to be a promising new tool for ranking DMUs. It keeps the idea of using DEA-based pairwise comparisons between the decision making units (DMUs), as was proposed originally in [38]. The basic new idea is to apply a variant of the CCR problem instead of the classic one. The ensuing scores are then utilized to build a nonreciprocal pairwise comparison matrix which serves as the basis of eliciting the ranking values of the DMUs. Main advantage of this new method is the wider range of the resulting ranking values which subsequently leads to better distinction between the DMUs. This useful property was also confirmed by numerical examples. Beside the eigenvector method, optimization based methods such as the logarithmic least squares method and the weighted least squares method were also tested for eliciting the ranking values from the nonreciprocal pairwise comparison matrix.

The numerical examples show that applying the new variant of the CCR problem and eliciting ranking weights from nonreciprocal pairwise comparison matrices remedy some shortcomings of the original method proposed in [38]. On the other hand, based upon the numerical examples, one cannot give a definite answer to the question of which of the techniques tested for eliciting ranking weights from nonreciprocal pairwise comparison matrices is the best. This question is argued but is undecided in the more general multiattribute decision making, too.

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