On the equivalency between a linear bilevel programming problem and linear optimization over the efficient set

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In the paper, the equivalency between a linear bilevel programming problem and linear optimization over the efficient set is investigated. As a consequence, we show that the problem of linear optimization over the efficient set is NP-hard.

bilevel programming; multiobjective programming; computational complexity

1. Introduction

In the paper we investigate the relationship between two special mathematical problems. The first one is a linear bilevel programming problem [6,7] stated as follows:

\[
\begin{align*}
\max_{x^1} & \quad c^{11T}x^1 + c^{12T}x^2, \text{ where } x^2 \text{ solves} \\
\max_{x^2} & \quad c^{21T}x^1 + c^{22T}x^2 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
Consider the mathematical programming problem

\[
\text{max } d^T x \text{ s.t. } x \in E(P),
\]

where \(d \in \mathbb{R}^n\). Problem (1.5) is a formulation of linear optimization over the efficient set and this is the second problem of our investigation. Linear optimization over the efficient set has several applications in multicriteria programming and is an important area of the current research in global optimization. See [3,4,5,8,10,14] for more details.

Attempts have been made to establish a relationship between the linear bilevel programming problem (1.1)-(1.3) and the linear bicriteria programming problem (1.4), where

\[
C = \begin{bmatrix}
c_{11}^T & c_{12}^T \\
c_{21}^T & c_{22}^T
\end{bmatrix}
\]

and \(P\) is the \((n_1 + n_2)\)-dimensional polyhedron of the points satisfying (1.3) \([1,15,16]\). Beside the theoretical interest, such relationship would be useful from computational point of view as well since there exist effective algorithms for bicriteria programming. It has been however recently demonstrated that, in general, there is no relationship between (1.1)-(1.3) and the bicriteria problem (1.4) constructed above. Moreover, given any two nonproportional vectors \(c_{12}\) and \(c_{22}\), a problem (1.1)-(1.3) can designed such that the optimal solutions of (1.1)-(1.3) are not efficient solutions of the bicriteria problem (1.4) defined above \([13]\).

In this paper, we make a new attempt to establish a relationship between the linear bilevel programming problem (1.1)-(1.3) and multicriteria programming. In Section 2, we show that a linear multicriteria programming problem can be constructed such that the feasible solutions of (1.1)-(1.3) coincide with the efficient solutions of the multicriteria programming problem. This later problem has \(r + 2\) criteria, where \(r = \text{rank } A_1\). Problem (1.1)-(1.3) can thus be reformulated as a problem (1.5). As another consequence, we point out that the problem of linear optimization over the efficient set is NP-hard.

In Section 3, the converse direction of the reformulation is investigated. We show that for any problem (1.5), a linear bilevel programming problem (1.1)-(1.3) can be constructed such that the optimal values of the problems are identical and there exists a simple correspondence between the optimal solutions of the two problems.

2. A multicriteria reformulation of the bilevel problem

A pair \((\bar{x}^1, \bar{x}^2)\), where \(\bar{x}^1 \in \mathbb{R}^{n_1}\) and \(\bar{x}^2 \in \mathbb{R}^{n_2}\), is a feasible solution of (1.1)-(1.3) when it fulfils (1.3) and \(\bar{x}^2\) is an optimal solution of

\[
\text{max } c_{21}^T \bar{x}^1 + c_{22}^T \bar{x}^2 \\
\text{s.t. } A_2 \bar{x}^2 \leq b - A_1 \bar{x}^1,
\]

where

(2.1)

(2.2)
where $\bar{x}^1$ is fixed in (2.1)-(2.2). Since $c^{21T}\bar{x}^1$ is only a constant term in the objective function (2.1), it may as well be omitted or $c^{21}$ can be simply replaced by a zero vector. A pair $(\bar{x}^1, \bar{x}^2)$ is an optimal solution of (1.1)-(1.3) when $(\bar{x}^1, \bar{x}^2)$ is a feasible solution of (1.1)-(1.3) and

$$c^{21T}\bar{x}^1 + c^{22T}\bar{x}^2 \geq c^{21T}x^1 + c^{22T}x^2$$

for every feasible solution $(x^1, x^2)$ of (1.1)-(1.3).

Given a problem (1.1)-(1.3), we define the multicriteria problem (1.4) as follows. Let $n = n_1 + n_2$, let $A = [A_1, A_2]$ be an $m \times n$ matrix, and let $P = \{ x \in \mathbb{R}^n \mid Ax \leq b \}$ (2.3) be a polyhedron in $\mathbb{R}^n$. Let $r = \text{rank } A_1$. Without loss of generality, we can assume that $A_1$ is decomposed as

$$A_1 = \begin{bmatrix} \tilde{A}_1 \\ \hat{A}_1 \end{bmatrix} ,$$

where $\tilde{A}_1$ is an $r \times n$ matrix and rank $\tilde{A}_1 = r$. Let $k = r + 2$ and let the $k \times n$ criterion matrix $C$ be defined by

$$C = \begin{bmatrix} \tilde{A}_1 & O \\ -e^T \hat{A}_1 & 0^1T \\ 0^1T & c^{22T} \end{bmatrix} ,$$

where $O$ is the $r \times n_2$ zero matrix, $0^1$ and $0^2$ are the $n_1$ and $n_2$-dimensional zero vectors, respectively, and $e$ is the $r$-vector with every entry equal to 1.

**Proposition 2.1:** A pair $(\bar{x}^1, \bar{x}^2)$ is a feasible solution of (1.1)-(1.3) if and only if

$$\bar{x} = \begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \end{bmatrix}$$

is an efficient solution of (1.4) defined by (2.3) and (2.4).

**Proof:** Assume that $(\bar{x}^1, \bar{x}^2)$ is a feasible solution of (1.1)-(1.3) but $\bar{x}$ is not efficient for (1.4). There exists then an $x \in P$ such that $Cx \geq C\bar{x}$ and $Cx \neq C\bar{x}$. Let $x$ be decomposed as $x = [x^{1T}, x^{2T}]^T$, where $x^1 \in \mathbb{R}^{n_1}$ and $x^2 \in \mathbb{R}^{n_2}$. Since $\tilde{A}_1x^1 \geq \tilde{A}_1\bar{x}^1$ and $-e^T \hat{A}_1x^1 \geq -e^T \hat{A}_1\bar{x}^1$, we get $\tilde{A}_1x^1 = \tilde{A}_1\bar{x}^1$ and $A_1x^1 = A_1\bar{x}^1$. In addition, from $A_1x^1 + A_2x^2 \leq b$, it follows that $x^2$ is a feasible solution of (2.1)-(2.2). Moreover, $Cx \neq C\bar{x}$ implies $c^{22T}x^2 > c^{22T}\bar{x}^2$. This later contradicts the optimality of $\bar{x}^2$ for (2.1)-(2.2) and the feasibility of $(\bar{x}^1, \bar{x}^2)$ for (1.1)-(1.3).

Conversely, assume that $\bar{x}$ is an efficient solution of (1.4) but $(\bar{x}^1, \bar{x}^2)$ is not a feasible solution of (1.1)-(1.3). The feasible set of (2.1)-(2.2) is not empty, e.g. $\bar{x}^2$ is a feasible solution. There exists
then an $\tilde{x}^2 \in R^{n_2}$ such that $\tilde{x}^2$ is feasible to (2.2) and $e^{22T} \tilde{x}^2 > c^{22T} \tilde{x}^2$. Let an $n$-vector $\tilde{x}$ be defined by

$$\tilde{x} = \begin{bmatrix} \tilde{x}^1 \\ \tilde{x}^2 \end{bmatrix}.$$ 

Clearly, $\tilde{x} \in P$, $C\tilde{x} \geq C\bar{x}$ and $C\tilde{x} \neq C\bar{x}$. This contradicts that $\bar{x}$ is an efficient solution of (1.4).

**Corollary 2.1:** A pair $(\bar{x}^1, \bar{x}^2)$ is an optimal solution of (1.1)-(1.3) if and only if

$$\bar{x} = \begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \end{bmatrix}$$

is an optimal solution of (1.5) defined by (2.3), (2.4) and

$$d = \begin{bmatrix} c^{11} \\ c^{12} \end{bmatrix}.$$

**Proposition 2.2:** Problem (1.5) is NP-hard.

**Proof:** The linear bilevel programming problem (1.1)-(1.3) is NP-hard [2,12]. As it has been shown above, (1.1)-(1.3) can be reduced to (1.5) through a polynomial transformation. The statement follows immediately.

3. A bilevel reformulation of linear programming over the efficient set

Consider a linear multicriteria programming problem (1.4), where $C$ is a $k \times n$ matrix, $P = \{x \in R^n \mid Ax \leq \bar{b}\}$, $A$ is an $m \times n$ matrix, $x \in R^n$ and $\bar{b} \in R^m$. For any $x \in P$, a linear programming problem

$$\max_y e^T C(y - x) \quad \text{s.t.} \quad Cy \geq Cx, \; y \in P \quad (3.1)$$

can be corresponded, where $e$ is the $k$-vector with every element equal to 1. It can be shown [9] that $x \in E(P)$ if and only if the optimal value of (3.1) is 0. In addition, independently whether $x$ is efficient, every optimal solution of (3.1) is efficient.

Consider the linear bilevel programming problem

$$\max_x d^T y, \; \text{where} \; y \; \text{solves} \quad (3.2)$$

$$\max_y - e^T Cx + e^T Cy \quad (3.3)$$

$$\text{s.t.} \; Ax \leq \bar{b}, \; Ay \leq \bar{b}, \; Cx - Cy \leq 0 \quad (3.4)$$

where $d \in R^n$. Denoting $n_1 = n_2 = n$, $m = 2m + k$, $x^1 = x$, $x^2 = y$, $c^{11} = 0$, $c^{12} = d$, $c^{21} = -e^T C$, $c^{22} = e^T C$,

$$A_1 = \begin{bmatrix} A \\ O \\ C \end{bmatrix}, \quad A_2 = \begin{bmatrix} O \\ A \\ -C \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} \bar{b} \\ \bar{b} \\ 0 \end{bmatrix},$$
where $O$ is the $m \times n$ zero matrix, problem (3.2)-(3.4) can be transcribed into the form (1.1)-(1.3).

**Proposition 3.1:** For an $\bar{x} \in \mathbb{R}^n$, the following statements are equivalent:

(a) $\bar{x}$ is an efficient solution of (1.4).

(b) The pair $(\bar{x}, \bar{x})$ is a feasible solution of (3.2)-(3.4).

(c) There exists an $x \in \mathbb{R}^n$ such that the pair $(x, \bar{x})$ is a feasible solution of (3.2)-(3.4).

**Proof:** (a) $\Rightarrow$ (b): Since $\bar{x}$ is an efficient solution of (1.4), the optimal value of problem (3.1) determined by $x = \bar{x}$ is 0 and $y = \bar{x}$ is an optimal solution. Consequently, if the leader chooses $x = \bar{x}$ in (3.2)-(3.4), then $y = \bar{x}$ is an optimal solution of the follower’s problem (3.3)-(3.4). The pair $(\bar{x}, \bar{x})$ is hence a feasible solution of (3.2)-(3.4).

(b) $\Rightarrow$ (c): It is evident with $x = \bar{x}$.

(c) $\Rightarrow$ (a): Since the pair $(x, \bar{x})$ is feasible to (3.2)-(3.4), $y = \bar{x}$ is an optimal solution of (3.3)-(3.4) when the leader has chosen $x$. Also, $y = \bar{x}$ is an optimal solution of problem (3.1) determined by $x$. By [9], $\bar{x}$ is an efficient solution of (1.4).

**Corollary 3.1:** Problem (1.4) has an efficient solution if and only if problem (3.2)-(3.4) has a feasible solution. Problem (1.5) has a finite optimal value if and only if problem (3.2)-(3.4) has a finite optimal value and the two optimal values are then identical. For an $\bar{x} \in \mathbb{R}^n$, the following statements are equivalent:

(a) $\bar{x}$ is an optimal solution of (1.5).

(b) The pair $(\bar{x}, \bar{x})$ is an optimal solution of (3.2)-(3.4).

(c) There exists an $x \in \mathbb{R}^n$ such that the pair $(x, \bar{x})$ is an optimal solution of (3.2)-(3.4).

**References**


