On the equivalency between a linear bilevel programming problem and linear optimization over the efficient set[0]

János FÜLÖP
Laboratory of Operations Research and Decision Systems,
Computer and Automation Institute, Hungarian Academy of Sciences,
H-1518 P.O.Box 63, Budapest, Hungary.

In the paper, the equivalency between a linear bilevel programming problem and linear optimization over the efficient set is investigated. As a consequence, we show that the problem of linear optimization over the efficient set is NP-hard.

bilevel programming; multiobjective programming; computational complexity

1. Introduction

In the paper we investigate the relationship between two special mathematical problems. The first one is a linear bilevel programming problem [6,7] stated as follows:

$$\max_{x^1} \ {c^{11}}^T x^1 + {c^{12}}^T x^2, \ \text{where } x^2 \text{ solves} \tag{1.1}$$

$$\max_{x^2} \ {c^{21}}^T x^1 + {c^{22}}^T x^2 \tag{1.2}$$

$$\text{s.t.} \ \ A_1 x^1 + A_2 x^2 \le b, \tag{1.3}$$

where $x^1, c^{11}, c^{12} \in R^{n_1}$, $x^2, c^{21}, c^{22} \in R^{n_2}$, $b \in R^m$, A_1 is an $m \times n_1$ matrix, A_2 is an $m \times n_2$ matrix, and ^{T} denotes the transposition. Such problems may arise when there are two decision makers at different hierarchical levels having joint constraints and different and possibly conflicting objectives. The decision-making process is sequential. The leader controlling the variables of x^1 has the first choice restricting thus the decision space of the follower, who controls the variables of x^2.

The second problem of the investigation connects with the multicriteria linear programming problem

$$\max \ Cx \ \text{ s.t. } \ x \in P, \tag{1.4}$$

where $x \in R^n$, C is a $k \times n$ matrix and $P \subseteq R^n$ is a polyhedron. Recall that a point $\bar{x} \in R^n$ is an efficient solution of (1.4) when $\bar{x} \in P$ and there exists no $x \in P$ such that $Cx \ge C\bar{x}$ and $Cx \ne C\bar{x}$. Let $E(P)$ denote the set of the efficient solutions of (1.4).

---

[0] *This research was supported in part by Hungarian National Research Foundation, OTKA No.2568.*

Consider the mathematical programming problem

$$\max\ d^T x\ \text{ s.t. }\ x \in E(P), \tag{1.5}$$

where $d \in R^n$. Problem (1.5) is a formulation of linear optimization over the efficient set and this is the second problem of our investigation. Linear optimization over the efficient set has several applications in multicriteria programming and is an important area of the current research in global optimization. See [3,4,5,8,10,14] for more details.

Attempts have been made to establish a relationship between the linear bilevel programming problem (1.1)-(1.3) and the linear bicriteria programming problem (1.4), where

$$C = \begin{bmatrix} {c^{11}}^T & {c^{12}}^T \\ {c^{21}}^T & {c^{22}}^T \end{bmatrix}$$

and P is the $(n_1 + n_2)$-dimensional polyhedron of the points satisfying (1.3) [1,15,16]. Beside the theoretical interest, such relationship would be useful from computational point of view as well since there exist effective algorithms for bicriteria programming. It has been however recently demonstrated that, in general, there is no relationship between (1.1)-(1.3) and the bicriteria problem (1.4) constructed above. Moreover, given any two nonproportional vectors c^{12} and c^{22}, a problem (1.1)-(1.3) can designed such that the optimal solutions of (1.1)-(1.3) are not efficient solutions of the bicriteria problem (1.4) defined above [13].

In this paper, we make a new attempt to establish a relationship between the linear bilevel programming problem (1.1)-(1.3) and multicriteria programming. In Section 2, we show that a linear multicriteria programming problem can be constructed such that the feasible solutions of (1.1)-(1.3) coincide with the efficient solutions of the multicriteria programming problem. This later problem has $r + 2$ criteria, where $r = \text{rank } A_1$. Problem (1.1)-(1.3) can thus be reformulated as a problem (1.5). As another consequence, we point out that the problem of linear optimization over the efficient set is NP-hard.

In Section 3, the converse direction of the reformulation is investigated. We show that for any problem (1.5), a linear bilevel programming problem (1.1)-(1.3) can be constructed such that the optimal values of the problems are identical and there exists a simple correspondence between the optimal solutions of the two problems.

2. A multicriteria reformulation of the bilevel problem

A pair $(\bar{x}^1, \bar{x}^2)$, where $\bar{x}^1 \in R^{n_1}$ and $\bar{x}^2 \in R^{n_2}$, is a feasible solution of (1.1)-(1.3) when it fulfils (1.3) and $\bar{x}^2$ is an optimal solution of

$$\max_{x^2}\ \ {c^{21}}^T \bar{x}^1 + {c^{22}}^T x^2 \tag{2.1}$$

$$\text{s.t.}\ \ A_2 x^2 \le b - A_1 \bar{x}^1, \tag{2.2}$$

where $\bar{x}^1$ is fixed in (2.1)-(2.2). Since ${c^{21}}^T \bar{x}^1$ is only a constant term in the objective function (2.1), it may as well be omitted or c^{21} can be simply replaced by a zero vector. A pair $(\bar{x}^1, \bar{x}^2)$ is an optimal solution of (1.1)-(1.3) when $(\bar{x}^1, \bar{x}^2)$ is a feasible solution of (1.1)-(1.3) and

$$ {c^{21}}^T \bar{x}^1 + {c^{22}}^T \bar{x}^2 \geq {c^{21}}^T x^1 + {c^{22}}^T x^2 $$

for every feasible solution (x^1, x^2) of (1.1)-(1.3).

Given a problem (1.1)-(1.3), we define the multicriteria problem (1.4) as follows. Let $n = n_1 + n_2$, let $A = [A_1, A_2]$ be an $m \times n$ matrix and let

$$ P = \{x \in R^n \mid Ax \leq b\} \tag{2.3} $$

be a polyhedron in R^n. Let $r = \text{rank } A_1$. Without loss of generality, we can assume that A_1 is decomposed as

$$ A_1 = \begin{bmatrix} \bar{A}_1 \\ \hat{A}_1 \end{bmatrix}, $$

where $\bar{A}_1$ is an $r \times n$ matrix and rank $\bar{A}_1 = r$. Let $k = r + 2$ and let the $k \times n$ criterion matrix C be defined by

$$ C = \begin{bmatrix} \bar{A}_1 & O \\ -e^T \bar{A}_1 & {0^2}^T \\ {0^1}^T & {c^{22}}^T \end{bmatrix}, \tag{2.4} $$

where O is the $r \times n_2$ zero matrix, 0^1 and 0^2 are the n_1 and n_2-dimensional zero vectors, respectively, and e is the r-vector with every entry equal to 1.

Proposition 2.1: *A pair $(\bar{x}^1, \bar{x}^2)$ is a feasible solution of (1.1)-(1.3) if and only if*

$$ \bar{x} = \begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \end{bmatrix} $$

is an efficient solution of (1.4) defined by (2.3) and (2.4).

Proof: Assume that $(\bar{x}^1, \bar{x}^2)$ is a feasible solution of (1.1)-(1.3) but $\bar{x}$ is not efficient for (1.4). There exists then an $x \in P$ such that $Cx \geq C\bar{x}$ and $Cx \neq C\bar{x}$. Let x be decomposed as $x = [{x^1}^T, {x^2}^T]^T$, where $x^1 \in R^{n_1}$ and $x^2 \in R^{n_2}$. Since $\bar{A}_1 x_1 \geq \bar{A}_1 \bar{x}_1$ and $-e^T \bar{A}_1 x_1 \geq -e^T \bar{A}_1 \bar{x}_1$, we get $\bar{A}_1 x_1 = \bar{A}_1 \bar{x}_1$ and $A_1 x_1 = A_1 \bar{x}_1$. In addition, from $A_1 x^1 + A_2 x^2 \leq b$, it follows that x^2 is a feasible solution of (2.1)-(2.2). Moreover, $Cx \neq C\bar{x}$ implies ${c^{22}}^T x^2 > {c^{22}}^T \bar{x}^2$. This later contradicts the optimality of $\bar{x}^2$ for (2.1)-(2.2) and the feasibility of $(\bar{x}^1, \bar{x}^2)$ for (1.1)-(1.3).

Conversely, assume that $\bar{x}$ is an efficient solution of (1.4) but $(\bar{x}^1, \bar{x}^2)$ is not a feasible solution of (1.1)-(1.3). The feasible set of (2.1)-(2.2) is not empty, e.g. $\bar{x}^2$ is a feasible solution. There exists then an $\tilde{x}^2 \in R^{n_2}$ such that $\tilde{x}^2$ is feasible to (2.2) and ${c^{22}}^T \tilde{x}^2 > {c^{22}}^T \bar{x}^2$. Let an n-vector $\tilde{x}$ be defined by

$$\tilde{x} = \begin{bmatrix} \bar{x}^1 \\ \tilde{x}^2 \end{bmatrix}.$$

Clearly, $\tilde{x} \in P$, $C\tilde{x} \geq C\bar{x}$ and $C\tilde{x} \neq C\bar{x}$. This contradicts that $\bar{x}$ is an efficient solution of (1.4). ∎

Corollary 2.1: *A pair $(\bar{x}^1, \bar{x}^2)$ is an optimal solution of (1.1)-(1.3) if and only if*

$$\bar{x} = \begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \end{bmatrix}$$

is an optimal solution of (1.5) defined by (2.3), (2.4) and

$$d = \begin{bmatrix} c^{11} \\ c^{12} \end{bmatrix}.$$

Proposition 2.2: *Problem (1.5) is NP-hard.*

Proof: The linear bilevel programming problem (1.1)-(1.3) is NP-hard [2,12]. As it has been shown above, (1.1)-(1.3) can be reduced to (1.5) through a polynomial transformation. The statement follows immediately. ∎

3. A bilevel reformulation of linear programming over the efficient set

Consider a linear multicriteria programming problem (1.4), where C is a $k \times n$ matrix, $P = \{x \in R^n \mid Ax \leq \bar{b}\}$, A is an $\bar{m} \times n$ matrix, $x \in R^n$ and $\bar{b} \in R^{\bar{m}}$. For any $x \in P$, a linear programming problem

$$\max_y \; e^T C(y - x) \;\text{ s.t. }\; Cy \geq Cx, \; y \in P \tag{3.1}$$

can be corresponded, where e is the k-vector with every element equal to 1. It can be shown [9] that $x \in E(P)$ if and only if the optimal value of (3.1) is 0. In addition, independently whether x is efficient, every optimal solution of (3.1) is efficient.

Consider the linear bilevel programming problem

$$\max_x \; d^T y, \;\text{ where } y \text{ solves} \tag{3.2}$$

$$\max_y \; -e^T Cx + e^T Cy \tag{3.3}$$

$$\text{s.t. } Ax \leq \bar{b}, \;\; Ay \leq \bar{b}, \;\; Cx - Cy \leq 0, \tag{3.4}$$

where $d \in R^n$. Denoting $n_1 = n_2 = n$, $m = 2\bar{m} + k$, $x^1 = x$, $x^2 = y$, $c^{11} = 0$, $c^{12} = d$, $c^{21} = -e^T C$, $c^{22} = e^T C$,

$$A_1 = \begin{bmatrix} A \\ O \\ C \end{bmatrix}, \qquad A_2 = \begin{bmatrix} O \\ A \\ -C \end{bmatrix} \qquad \text{and} \qquad b = \begin{bmatrix} \bar{b} \\ \bar{b} \\ 0 \end{bmatrix},$$

where O is the $\bar{m} \times n$ zero matrix, problem (3.2)-(3.4) can be transcribed into the form (1.1)-(1.3).

Proposition 3.1: *For an $\bar{x} \in R^n$, the following statements are equivalent:*

(a) $\bar{x}$ is an efficient solution of (1.4).

(b) The pair $(\bar{x}, \bar{x})$ is a feasible solution of (3.2)-(3.4).

(c) There exists an $x \in R^n$ such that the pair $(x, \bar{x})$ is a feasible solution of (3.2)-(3.4).

Proof: (a) $\Rightarrow$ (b): Since $\bar{x}$ is an efficient solution of (1.4), the optimal value of problem (3.1) determined by $x = \bar{x}$ is 0 and $y = \bar{x}$ is an optimal solution. Consequently, if the leader chooses $x = \bar{x}$ in (3.2)-(3.4), then $y = \bar{x}$ is an optimal solution of the follower's problem (3.3)-(3.4). The pair $(\bar{x}, \bar{x})$ is hence a feasible solution of (3.2)-(3.4).

(b) $\Rightarrow$ (c): It is evident with $x = \bar{x}$.

(c) $\Rightarrow$ (a): Since the pair $(x, \bar{x})$ is feasible to (3.2)-(3.4), $y = \bar{x}$ is an optimal solution of (3.3)-(3.4) when the leader has chosen x. Also, $y = \bar{x}$ is an optimal solution of problem (3.1) determined by x. By [9], $\bar{x}$ is an efficient solution of (1.4). ∎

Corollary 3.1: *Problem (1.4) has an efficient solution if and only if problem (3.2)-(3.4) has a feasible solution. Problem (1.5) has a finite optimal value if and only if problem (3.2)-(3.4) has a finite optimal value and the two optimal values are then identical. For an $\bar{x} \in R^n$, the following statements are equivalent:*

(a) $\bar{x}$ is an optimal solution of (1.5).

(b) The pair $(\bar{x}, \bar{x})$ is an optimal solution of (3.2)-(3.4).

(c) There exists an $x \in R^n$ such that the pair $(x, \bar{x})$ is an optimal solution of (3.2)-(3.4).

References

[1] J.F. Bard, "An efficient point algorithm for a linear two-stage optimization problem", *Operations Research* **31**, 670-684 (1983).

[2] J.F. Bard, "Some properties of the bilevel programming problem", *Journal of Optimization Theory and Applications* **68**, 371-378 (1991).

[3] H.P. Benson, "Optimization over the efficient set", *Journal of Mathematical Analysis and Applications* **98**, 562-580 (1984).

[4] H.P. Benson, "An all-linear programming relaxation algorithm for optimizing over the efficient set", *Journal of Global Optimization* **1**, 83-104 (1991).

[5] H.P. Benson, (1992), "A finite, nonadjacent extreme-point search algorithm for optimizing over the efficient set", *Journal of Optimization Theory and Applications* **73**, 47-64 (1992).

[6] W.F. Bialas and M.H. Karwan, "Two-level linear programming", *Management Science* **30**, 1004-1020 (1984).

[7] W. Candler and R. Townsley, "A linear two-level programming problem", *Computers and Operations Research* **9**, 59-76 (1982).

[8] J.P. Dauer, "Optimization over the efficient set using an active constraint approach", *ZOR – Methods and Models of Operations Research* **35**, 185-195 (1991).

[9] J.G. Ecker and I.A. Kouada, "Finding efficient points for linear multiple objective programs", *Mathematical Programming* **8**, 375-377 (1975).

[10] J. Fülöp, "A cutting plane method for linear optimization over the efficient set", Working Paper 92-15, Laboratory of Operations Research and Decision Systems, Computer and Automation Institute, Hungarian Academy of Sciences, Budapest, October 1992.

[11] M.R. Garey and D.S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W.H. Freeman and Company, New York, 1979.

[12] R.G. Jeroslow, "The polynomial hierarchy and a simple model for competitive analysis", *Mathematical Programming* **32**, 146-164 (1985).

[13] P. Marcotte and G. Savard, "A note on the Pareto optimality of solutions to the linear bilevel programming problem", *Computers and Operations Research* **18**, 355-359 (1991).

[14] J. Philip, "Algorithms for the vector maximization problem", *Mathematical Programming* **2**, 207-229 (1972).

[15] G. Ünlü, "A linear bilevel programming algorithm based on bicriteria programming", *Computers and Operations Research* **14**, 173-179 (1987).

[16] U.-P. Wen and S.-T. Hsu, "A note on a linear bilevel programming algorithm based on bicriteria programming", *Computers and Operations Research* **16**, 79-83 (1989).