ON ABRAMS' THEOREM

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On Abrams' theorem

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Abstract. Abrams' theorem describes a necessary and sufficient condition for the closedness of a linear image of an arbitrary set. Closedness conditions of this type play an important role in the theory of duality in convex programming. In this paper we present generalizations of Abrams' theorem, as well as Abrams-type theorems characterizing other properties (such as relatively openness or polyhedrality) of linear images of convex sets.

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1 Introduction

Closedness conditions play an important role in the theory of duality in convex programming. Conditions which imply the closedness of convex sets of the form

$$(AC_1) + C_2 = \{Ax + y : x \in C_1,\, y \in C_2\}, \tag{1}$$

where $A \in \mathcal{R}^{m \times n}$ is a matrix, $C_1 \subseteq \mathcal{R}^n$ and $C_2 \subseteq \mathcal{R}^m$ are convex sets, are particularly useful, see for example [9]. Before stating the main result of [9], we fix some notation.

Let us denote by $\operatorname{rec} C$ and $\operatorname{bar} C$ the *recession cone* and the *barrier cone* of a convex set C in $\mathcal{R}^d$, that is let

$$\operatorname{rec} C := \left\{ z \in \mathcal{R}^d : x + \lambda z \in C\ (x \in C,\, \lambda \geq 0) \right\},$$
$$\operatorname{bar} C := \left\{ a \in \mathcal{R}^d : \inf\{a^T x : x \in C\} > -\infty \right\}.$$

Then $\operatorname{rec} C$ and $\operatorname{bar} C$ are convex cones. Furthermore, if C is a nonempty closed convex set in $\mathcal{R}^d$ then $\operatorname{rec} C$ is the dual cone of $\operatorname{bar} C$. The barrier cone and thus, dually, the recession cone also are not changed via adding of a compact convex set to a closed convex set.

Let us denote by $\operatorname{ri} C$ ($\operatorname{cl} C$) the *relative interior* (*closure*) of the convex set C in $\mathcal{R}^d$. The relative interior of a convex set C is convex, and is nonempty if the convex set C is nonempty. (See [3] for the definition and properties of the relative interior.)

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The main result of [9] describes two equivalent sufficient conditions for the closedness of the set (1) where C_1 is a closed convex set and $C_2 = P_2$ is a polyhedron:

THEOREM 1.1. *Let A be an m by n real matrix, let P_2 be a polyhedron in $\mathcal{R}^m$, and let C_1 be a closed convex set in $\mathcal{R}^n$. Then between the statements*

a) $(A^T \mathrm{bar}\, P_2) \cap \mathrm{ri}\,(\mathrm{bar}\, C_1) \neq \emptyset$,
b) $A^{-1}(-\mathrm{rec}\, P_2) \cap (\mathrm{rec}\, C_1) \subseteq -\mathrm{rec}\, C_1$,
c) $(AC_1) + P_2$ is closed, and $\mathrm{rec}\,((AC_1) + P_2) = (A\mathrm{rec}\, C_1) + \mathrm{rec}\, P_2$,

hold the following logical relations: a) is equivalent to b); a) or b) implies c).

In [9] two proofs are given for Theorem 1.1, the second one is of a more algebraic nature than the first one: it does not use the Bolzano-Weierstrass theorem. This proof hinges on the following key fact from [1]:

THEOREM 1.2. (Abrams) *Let A be an m by n real matrix, and let S_1 be an arbitrary set in $\mathcal{R}^n$. Then the set AS_1 is closed if and only if the set $S_1 + A^{-1}(\{0\})$ is closed.*

In this paper our aim is to generalize Abrams' theorem and derive further sufficient conditions for the closedness of convex sets of the form (1). In Section 2 we prove Abrams-type theorems, formally obtained by exchanging the closedness property in Abrams' theorem for the properties relatively open, has only finitely many faces, and polyhedral, respectively. In Section 3 we derive a lemma that will be applied in Section 4 in the proof of a generalized Abrams' theorem. The lemma states that if we add a polytope to a convex set, and the resulting set is closed then the original convex set is closed. Here also, similar statements hold if we exchange the closedness property for the properties relatively open, has only finitely many exposed faces, and polyhedral, respectively. Finally, in Section 4 we present three generalizations of Abrams' theorem, and an equivalent form of statement c) in Theorem 1.1.

2 Abrams-type theorems

In this section we prove three theorems similar to Abrams' theorem. These theorems can be formally obtained by replacing the closedness conditions in Abrams' theorem by the assumption that the sets under consideration are relatively open, have only finitely many faces, and are polyhedrons, respectively.

First, note that

$$A^{-1}(AS_1) = S_1 + A^{-1}(\{0\}) \text{ and } A(S_1 + A^{-1}(\{0\})) = AS_1 \qquad (2)$$

for any matrix $A \in \mathcal{R}^{m \times n}$ and any set $S_1 \subseteq \mathcal{R}^n$. (Here $A^{-1}(S_2) := \{x : Ax \in S_2\}$ for any set S_2 in $\mathcal{R}^m$.) Hence, to prove an Abrams-type theorem it is enough

to prove that if a convex set has the corresponding property (is relatively open,... etc.) then its linear image and inverse image have this property also.

In the case of the relatively open convex sets (i.e. convex sets C with $C = \text{ri}\,C$) this fact is an immediate consequence of the following lemma (Theorems 6.6 and 6.7 in [3]):

LEMMA 2.1. *Let A be an m by n real matrix. Let C_1 be a convex set in $\mathcal{R}^n$, and let C_2 be a convex set in $\mathcal{R}^m$ such that $A^{-1}(\text{ri}\,C_2) \neq \emptyset$. Then*

- $\text{ri}(AC_1) = A(\text{ri}\,C_1)$;

- $\text{ri}\,(A^{-1}C_2) = A^{-1}(\text{ri}\,C_2)$.

By Lemma 2.1 and (2) we have

THEOREM 2.1. *Let A be an m by n real matrix, and let C_1 be a convex set in $\mathcal{R}^n$. Then the convex set AC_1 is relatively open if and only if the convex set $C_1 + A^{-1}(\{0\})$ is relatively open.* □

A convex set $F \subseteq \mathcal{R}^d$ is called the *face* of the convex set $C \subseteq \mathcal{R}^d$, if $F \subseteq C$, and for every $x_1, x_2 \in C$, $0 < \varepsilon < 1$, $\varepsilon x_1 + (1-\varepsilon)x_2 \in F$ implies that $x_1, x_2 \in F$.

The following lemma can be found in [3], see Theorem 18.2:

LEMMA 2.2. *Let C be a non-empty convex set, and let U be the collection of all the relative interiors of non-empty faces of C. Then U is a partition of C, i.e. the sets in U are disjoint and their union is C. Every relatively open convex subset of C is contained in one of the sets in U, and these are the maximal relatively open convex subsets of C.*

From Lemma 2.2 it can be easily seen that a convex set C has only finitely many faces if and only if C is the union of finitely many relatively open convex sets. From this observation and Lemma 2.1 readily follows

THEOREM 2.2. *Let A be an m by n real matrix, and let C_1 be a convex set in $\mathcal{R}^n$. Then the convex set AC_1 has only finitely many faces if and only if the convex set $C_1 + A^{-1}(\{0\})$ has only finitely many faces.* □

Finally, in the case of polyhedral convex sets we need Motzkin's theorem ([5]) to prove that the linear image of a polyhedron is also a polyhedron (for inverse images the statement is obvious).

A convex set $P_1 \subseteq \mathcal{R}^n$ is called a *polyhedron* if there exist a matrix $A \in \mathcal{R}^{m \times n}$ and a vector $b \in \mathcal{R}^m$ such that $P_1 = \{x : Ax \leq b\}$. A convex set $Q_1 \subseteq \mathcal{R}^n$ $[R_1 \subseteq \mathcal{R}^n]$ is called a *polytope* [*finitely generated cone*] if there exists a finite set $S_1 \subseteq \mathcal{R}^n$ such that Q_1 $[R_1]$ is the convex hull [convex conical hull] of S_1.

LEMMA 2.3. (Motzkin) *For every polyhedron P there exist a polytope Q and a finitely generated cone R such that $P = Q + R$. Conversely, the sum of a polytope and a finitely generated cone is a polyhedron.*

Hence, linear images of polyhedrons are sums of linear images of polytopes (polytopes also) and linear images of finitely generated cones (finitely generated cones also). Again by Motzkin's theorem they are polyhedrons. Using this observation and (2) the following Abrams-type theorem can be easily verified.

THEOREM 2.3. *Let A be an m by n real matrix, and let C_1 be a convex set in $\mathcal{R}^n$. Then the convex set AC_1 is a polyhedron if and only if the convex set $C_1 + A^{-1}(\{0\})$ is a polyhedron.* $\square$

3 Adding a polytope

In this section we will study the following type of question: if we add a polytope to a convex set, and the sum has a certain property (closed, relatively open,... etc.), then does the original convex set have this property also? These questions are only seemingly trivial, as we will see the proofs use involved results from linear algebra.

The proofs of Theorems 3.1, 3.2 and 3.4 are based on the following lemma.

LEMMA 3.1. *Let $Q \subseteq \mathcal{R}^d$ be the convex hull of the points $q_1, \ldots, q_t$. Let $p_i \in Q - \{q_i\}$ for $i = 1, \ldots, t$. Then there exist nonnegative not all zero constants $\lambda_1, \ldots, \lambda_t$ such that $\sum_{i=1}^{t} \lambda_i p_i = 0$.*

Proof. The points p_i are of the form

$$p_i = \sum_{j=1}^{t} \varepsilon_{ij} q_j - q_i \qquad (i = 1, \ldots, t),$$

where $\varepsilon_{ij} \geq 0$ for all i, j, and $\sum_{j=1}^{t} \varepsilon_{ij} = 1$ for all i. Let E denote the t by t matrix with (i, j)-th element ε_{ij}. Then E is an elementwise nonnegative matrix with row sums equal to 1. It can be easily seen that the absolute value of each eigenvalue of the matrix E is at most 1. Really, if ε is an eigenvalue of the matrix E, then there exists a nonzero vector v such that $Ev = \varepsilon v$. Let v_i have the largest absolute value amongst the elements of the vector v. Then

$$|\varepsilon| \cdot |v_i| = \left| \sum_{j=1}^{t} \varepsilon_{ij} v_j \right| \leq |v_i| \cdot \sum_{j=1}^{t} \varepsilon_{ij} = |v_i|$$

holds, and we have $|\varepsilon| \leq 1$.

Hence, $\varepsilon = 1$ is an eigenvalue of the matrix E with maximal absolute value. As the matrices E and E^T are similar (see Exercise B.4 in [6]), the spectrums of E and E^T are the same. Thus $\varepsilon = 1$ is an eigenvalue of the nonnegative matrix E^T with maximal absolute value, also. By the Perron-Frobenius theorem

(Theorem 9.2.1 in [4]) there exists a nonnegative eigenvector of the matrix E^T with corresponding eigenvalue 1.

Let us denote by $\lambda_1, \ldots, \lambda_t$ the elements of this eigenvector. Then the numbers λ_i $(i = 1, \ldots, t)$ meet the requirements:

$$
\begin{aligned}
\sum_{i=1}^{t} \lambda_i p_i &= \sum_{i=1}^{t}\sum_{j=1}^{t} \lambda_i \varepsilon_{ij} q_j - \sum_{i=1}^{t} \lambda_i q_i \\
&= \sum_{j=1}^{t} \left(\sum_{i=1}^{t} \lambda_i \varepsilon_{ij} - \lambda_j \right) q_j \\
&= \sum_{j=1}^{t} 0 \cdot q_j = 0
\end{aligned}
$$

holds; and the lemma is proved. $\square$

Now, Lemma 3.1 can be applied to derive

THEOREM 3.1. *Let $Q \subseteq \mathcal{R}^d$ be the convex hull of the points $q_1, \ldots, q_t$, and let $C \subseteq \mathcal{R}^d$ be a convex set. If $Q + C$ is closed then C is closed also.*

Proof. Let $c_i \in C$ $(i = 1, 2, \ldots)$, and suppose that $c_i \to c_\infty$ $(i \to \infty)$. We have to show that $c_\infty \in C$. For every j the sequence $q_j + c_i$ $(i = 1, 2, \ldots)$ converges to $q_j + c_\infty$. As the set $Q + C$ is closed, so the limit point is in this set. Hence there exist $q_j' \in Q$ and $c_j' \in C$ such that $q_j + c_\infty = q_j' + c_j'$ $(j = 1, \ldots, t)$. By Lemma 3.1 there exist nonnegative not all zero constants $\lambda_1, \ldots, \lambda_t$ such that $\sum_{j=1}^{t} \lambda_j (q_j' - q_j) = 0$. But then

$$
\sum_{j=1}^{t} \lambda_j c_\infty = \sum_{j=1}^{t} \lambda_j c_j' + \sum_{j=1}^{t} \lambda_j (q_j' - q_j) = \sum_{j=1}^{t} \lambda_j c_j'
$$

holds; and the vector c_∞ being a convex combination of the vectors $c_j' \in C$, is in C also. Thus the set C is closed, which was to be shown. $\square$

We remark that the converse of Theorem 3.1 (closedness of C implies the closedness of $Q + C$) follows by the Bolzano-Weierstrass theorem. (For another proof, without using the Bolzano-Weierstrass theorem, see [9].)

THEOREM 3.2. *Let $Q \subseteq \mathcal{R}^d$ be the convex hull of the points $q_1, \ldots, q_t$, and let $C \subseteq \mathcal{R}^d$ be a convex set. If $Q + C$ is relatively open, then C is relatively open also.*

Proof. The set $Q + C$ is relatively open, which means that

$$
Q + C = (\operatorname{ri} Q) + (\operatorname{ri} C).
$$

Let $c \in C$, then for every $i = 1, \ldots, t$ there exist $q_i' \in \operatorname{ri} Q$ and $c_i' \in \operatorname{ri} C$ such that $q_i + c = q_i' + c_i'$. By Lemma 3.1 there exist nonnegative not all zero constants $\lambda_1, \ldots, \lambda_t$ such that $\sum_{i=1}^{t} \lambda_i(q_i' - q_i) = 0$. Similarly as in the proof of Theorem 3.1, we can see that c is a convex combination of the vectors $c_i' \in \operatorname{ri} C$; and so by convexity of $\operatorname{ri} C$, $c \in \operatorname{ri} C$ also. We have proved the inclusion $C \subseteq \operatorname{ri} C$. As the other inclusion $\operatorname{ri} C \subseteq C$ is trivial, we have $C = \operatorname{ri} C$, and the proof is finished. $\square$

Let $C \subseteq \mathcal{R}^d$ be a convex set. If for an F subset of C there exist a vector $a \in \mathcal{R}^d$ and a constant $\beta \in \mathcal{R}$ such that

$$a^T f = \beta < a^T c \qquad (c \in C \setminus F,\, f \in F),$$

then F is called an *exposed subset* of the set C. Every exposed subset is a face, but generally not vice versa.

THEOREM 3.3.   *Let $Q \subseteq \mathcal{R}^d$ be a polytope, and let $C \subseteq \mathcal{R}^d$ be a convex set. If the convex set $Q + C$ has only finitely many exposed subsets, then the convex set C has only finitely many exposed subsets also.*

Proof. Let S denote the set of pairs (F, G), where $F \subseteq C$, $G \subseteq Q$, and there exist a vector $a \in \mathcal{R}^d$ and constants $\beta, \gamma \in \mathcal{R}$ such that

$$a^T f = \beta < a^T c \quad (f \in F,\, c \in C \setminus F), a^T g = \gamma < a^T q \quad (g \in G,\, q \in Q \setminus G).$$

It is easy to see that if $(F, G) \in S$ then $F + G$ is an exposed subset of $Q + C$. On the other hand, for every $(F, G) \in S$, the equations

$$F = C \cap (F + G - Q),\, G = Q \cap (F + G - C)$$

can be easily verified. Hence the map $(F, G) \mapsto F + G$ is injective on S, and maps S into the finite set of exposed subsets of $Q + C$. Consequently, S has only finitely many elements. For every exposed subset F of C, there exists an exposed subset G of Q such that $(F, G) \in S$. Thus the number of exposed subsets of C is finite, which was to be shown. $\square$

With minor modification of the proof of Theorem 19.1 in [3], it can be proved that a convex set is a polyhedron if and only if it is closed and has only finitely many exposed subsets. (A possible proof uses Theorems 18.7 and 19.6 instead of Theorem 18.5. See also [7], Theorem 7.12.) Hence the following theorem is an immediate consequence of Theorems 3.1 and 3.3.

THEOREM 3.4.   *Let $Q \subseteq \mathcal{R}^d$ be a polytope, and let $C \subseteq \mathcal{R}^d$ be a convex set. If $Q + C$ is a polyhedron, then C is a polyhedron also.* $\square$

4 Generalizations of Abrams' theorem

In this section we describe three generalizations of Abrams' theorem. The relation between the closedness of the sets $(AS_1) + S_2$ and $S_1 + A^{-1}(S_2)$ is studied, first for arbitrary sets S_1, S_2, then with additional assumptions on the set S_2.

Let A be an m by n real matrix. Then there exists a unique n by m matrix X such that: a) $AXA = A$; b) $XAX = X$; c) $(AX)^T = AX$; d) $(XA)^T = XA$. This matrix X is called the *Moore-Penrose generalized inverse* of the matrix A, and is denoted by $A^\dagger$. It is well-known that then the matrix $AA^\dagger$ is the matrix of the orthogonal projection to the subspace $A\mathcal{R}^n$ (see [4], [6]).

We adapt the technique of the original proof of Abrams' theorem (see [1]) to prove the following generalized Abrams' theorem.

THEOREM 4.1. *Let $A \in \mathcal{R}^{m \times n}$ be a matrix. Let S_1 be a set in $\mathcal{R}^n$, and let S_2 be a set in $\mathcal{R}^m$. Then the closedness of the set $(AS_1) + S_2$ implies the closedness of the set $S_1 + A^{-1}(S_2)$. Conversely also, if $S_2 \subseteq A\mathcal{R}^n$ holds.*

Proof. First, we will show that if the set $(AS_1) + S_2$ is closed, then the set $S_1 + A^{-1}(S_2)$ is closed also. Let $x_i \in S_1$, $v_i \in A^{-1}(S_2)$ $(i = 1, 2, \ldots)$, and suppose that
$$x_i + v_i \to z \ (i \to \infty).$$
Then
$$Ax_i + Av_i \to Az \ (i \to \infty),$$
and $Az \in (AS_1) + S_2$ because of the closedness of the set $(AS_1) + S_2$. Hence there exist points $x \in S_1$, $y \in S_2$ such that $Az = Ax + y$. Then
$$z = x + (z - x) \in S_1 + A^{-1}(S_2),$$
which was to be shown.

To prove the converse direction, let $x_i \in S_1$, and let $y_i \in S_2$ $(i = 1, 2, \ldots)$. Suppose that
$$Ax_i + y_i \to b \ (i \to \infty).$$
Then by the assumption that $S_2 \subseteq A\mathcal{R}^n$, we have $b \in A\mathcal{R}^n$. Furthermore,
$$A^\dagger Ax_i + A^\dagger y_i \to A^\dagger b \ (i \to \infty),$$
which can be written as
$$x_i + v_i \to A^\dagger b \ (i \to \infty)$$
where v_i denotes the following vector:
$$v_i := A^\dagger y_i - x_i + A^\dagger Ax_i \ (i = 1, 2, \ldots).$$

The assumption that $S_2 \subseteq A\mathcal{R}^n$ implies that $v_i \in A^{-1}(S_2)$ $(i = 1, 2, \ldots)$. Thus by the closedness of the set $S_1 + A^{-1}(S_2)$, we have $A^\dagger b \in S_1 + A^{-1}(S_2)$. Hence $b = AA^\dagger b \in (AS_1) + S_2$, which was to be shown. $\square$

Let A be the matrix $0 \in \mathcal{R}^{1 \times 1}$, let $S_1 \subseteq \mathcal{R}$ be an arbitrary set, and let $S_2 \subseteq \mathcal{R}$ be a not closed set such that $0 \in S_2$. This example shows that generally the assumption that $S_2 \subseteq A\mathcal{R}^n$ is needed in the theorem. However this is not the case when S_2 is a subspace, as the following theorem shows.

THEOREM 4.2. *Let A be an m by n real matrix. Let S_1 be a set in $\mathcal{R}^n$, and let L_2 be a subspace in $\mathcal{R}^m$. Then the set $(AS_1) + L_2$ is closed if and only if the set $S_1 + A^{-1}(L_2)$ is closed.*

Proof. Choose a matrix B such that $L_2 = B^{-1}(\{0\})$. Then by Abrams' theorem the set $(AS_1) + L_2$ is closed if and only if the set BAS_1 is closed. On the other hand, $A^{-1}(L_2) = (BA)^{-1}(\{0\})$ holds, so again by Abrams' theorem the set $S_1 + A^{-1}(L_2)$ is closed if and only if the set BAS_1 is closed. Hence the theorem is proved. $\square$

It is an open problem whether Theorem 4.2 holds with a polyhedron P_2 instead of the subspace L_2. We were able to prove this conjecture only in the case when $S_1 = C_1$ is a convex set:

THEOREM 4.3. *Let A be an m by n real matrix. Let C_1 be a convex set in $\mathcal{R}^n$, and let P_2 be a polyhedron in $\mathcal{R}^m$. Then the set $(AC_1) + P_2$ is closed if and only if the set $C_1 + A^{-1}(P_2)$ is closed.*

Proof. We have to prove only that the closedness of the set $C_1 + A^{-1}(P_2)$ implies the closedness of the set $AC_1 + P_2$ (the other direction is obvious).

Let us suppose first that $P_2 = R_2$ is a polyhedral cone. As $A^{-1}(R_2) = A^{-1}(R_2 \cap A\mathcal{R}^n)$, by Theorem 4.1 the closedness of the set $C_1 + A^{-1}(R_2)$ implies the closedness of the set $AC_1 + R_2 \cap A\mathcal{R}^n$. We can apply Theorem 1.1 to show that then the set $AC_1 + R_2$ is closed also. Obviously,

$$
\begin{aligned}
(-R_2) \cap \operatorname{rec}(AC_1 + R_2 \cap A\mathcal{R}^n) &\subseteq -(R_2 \cap A\mathcal{R}^n) \\
&\subseteq -\operatorname{rec}(AC_1 + R_2 \cap A\mathcal{R}^n).
\end{aligned}
$$

By Theorem 1.1, the sum of the closed convex sets $AC_1 + R_2 \cap A\mathcal{R}^n$ and R_2 is closed, that is the set $AC_1 + R_2$ is closed. This finishes the proof in the special case when $P_2 = R_2$ is a polyhedral cone.

For the general case let us suppose that the set $C_1 + A^{-1}(P_2)$ is closed. The set $A^{-1}(P_2)$ is a polyhedron, with recession cone $A^{-1}(\operatorname{rec} P_2)$. By Motzkin's theorem $A^{-1}(P_2)$ is the sum of a polytope and the polyhedral cone $A^{-1}(\operatorname{rec} P_2)$. Thus by Theorem 3.1 the closedness of the set $C_1 + A^{-1}(P_2)$ implies the closedness of the set $C_1 + A^{-1}(\operatorname{rec} P_2)$, which in turn, as we have seen it in the first half of the proof, implies the closedness of the set $AC_1 + \operatorname{rec} P_2$. By Motzkin's

theorem the polyhedron P_2 is the sum of a polytope Q_2 and the polyhedral cone $\operatorname{rec} P_2$, so by compactness of the polytope Q_2 the closedness of the set $AC_1 + P_2$ follows. This way we have proved the nontrivial implication in the theorem in the general case as well. $\square$

As we have seen it in the first proof of Theorem 1.3 in [9] (Theorem 1.1 in this paper), after showing the closedness of the set $(AC_1) + P_2$, further steps (Steps 6 and 7) are needed to prove the part concerning the recession cones. Similarly, in the case of Theorem 4.3, this extension is also possible, see Theorem 4.4 below. The proof of Theorem 4.4 is analogous to the above-mentioned proof; so we only describe a lemma here, and the details are left to the reader.

Let $\mathcal{K}(C)$ denote the convex conical hull of the set $\{1\} \times C$, for any convex set C in $\mathcal{R}^d$. We mention here only two relevant properties of the $\mathcal{K}(.)$ operation (see [3]): the intersection of the cone $\operatorname{cl} \mathcal{K}(C)$ and the hyperplane $\{1\} \times \mathcal{R}^d$ is $\{1\} \times (\operatorname{cl} C)$, while the intersection of the cone $\operatorname{cl} \mathcal{K}(C)$ and the hyperplane $\{0\} \times \mathcal{R}^d$ is $\{0\} \times \operatorname{rec}(\operatorname{cl} C)$.

LEMMA 4.1. *Let A be an m by n real matrix. Let C_1 be a closed convex set in $\mathcal{R}^n$, and let R_2 be a polyhedral cone in $\mathcal{R}^m$. Let us denote by $\hat{K}_1$ and $\hat{K}_2$ the following two convex cones in $\mathcal{R}^{n+1}$ and in $\mathcal{R}^{m+1}$, respectively:*

$$\hat{K}_1 \quad := \quad (\operatorname{cl} \mathcal{K}(C_1)) + \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ R_2 \end{pmatrix};$$

$$\hat{K}_2 \quad := \quad \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \operatorname{cl} \mathcal{K}(C_1) + \begin{pmatrix} 0 \\ R_2 \end{pmatrix}.$$

With this notation the following statements are equivalent:
a) $\hat{K}_2$ is closed;
b) $\hat{K}_2 = \operatorname{cl} \mathcal{K}(AC_1 + R_2)$;
c) $AC_1 + R_2$ is closed, and $\operatorname{rec}(AC_1 + R_2) = (A \operatorname{rec} C_1) + R_2$;
d) $\hat{K}_1$ is closed;
e) $\hat{K}_1 = \operatorname{cl} \mathcal{K}(C_1 + A^{-1}(R_2))$;
f) $C_1 + A^{-1}(R_2)$ is closed, and $\operatorname{rec}(C_1 + A^{-1}(R_2)) = (\operatorname{rec} C_1) + A^{-1}(R_2)$.

Proof. a)$\Leftrightarrow$b): We have proved in Step 6 of the first proof of Theorem 1.3 in [9], that the closure of the cone $\hat{K}_2$ is $\operatorname{cl} \mathcal{K}(AC_1 + R_2)$.

b)$\Leftrightarrow$c): The two convex cones $\hat{K}_2$ and $\operatorname{cl} \mathcal{K}(AC_1 + R_2)$ are equal if and only if their intersection with the hyperplanes $\{1\} \times \mathcal{R}^m$ and $\{0\} \times \mathcal{R}^m$ are equal.

d)$\Leftrightarrow$e) and e)$\Leftrightarrow$f) can be proved analogously.

Finally, a)$\Leftrightarrow$d) is a consequence of Theorem 4.3. $\square$

Now, applying Lemma 4.1, Theorem 3.1 and its converse, we can easily derive

THEOREM 4.4. *Let A be an m by n real matrix. Let C_1 be a closed convex set in $\mathcal{R}^n$, and let P_2 be a polyhedron in $\mathcal{R}^m$. Then the following statements are equivalent:*

a) $(AC_1) + P_2$ *is closed, and* $\operatorname{rec}((AC_1) + P_2) = (A \operatorname{rec} C_1) + \operatorname{rec} P_2$*;*

b) $C_1 + A^{-1}(P_2)$ *is closed, and* $\operatorname{rec}(C_1 + A^{-1}(P_2)) = (\operatorname{rec} C_1) + A^{-1}(\operatorname{rec} P_2)$. $\square$

Note that statement a) in Theorem 4.4 is exactly statement c) in Theorem 1.1.

Finally, we remark that Theorem 4.3 (and Theorem 4.4) can not be generalized to closed convex sets instead of polyhedrons: There exist closed convex sets (cones) C_1 and C_2 such that the set $C_1 + A^{-1}(C_2)$ is closed but the set $(AC_1) + C_2$ is not closed. Really, let

$$A : x \mapsto x \cdot J \quad (x \in \mathcal{R}), \, C_1 := \mathcal{R}, \, C_2 := \text{PSD}$$

where $J \in \mathcal{R}^{n \times n}$ denotes the matrix with all elements equal to one, and PSD denotes the closed convex cone of the n by n real symmetric positive semidefinite matrices. It is proved in [8] (see Proposition 1.1) that the set $(AC_1) + C_2$ is not closed. On the other hand, the set $C_1 + A^{-1}(C_2)$ is closed; so A, C_1 and C_2 indeed meet the requirements.

Conclusion. Abrams' theorem characterizes the closedness of a linear image of an arbitrary set. In this paper we described three generalizations of Abrams' theorem. Also we presented Abrams-type theorems characterizing the relatively openness, finitely (exposed) facedness and polyhedrality of the linear image of a convex set, and the Minkowski sum of a polytope and a convex set.

References

1. BERMAN, A., *Cones, matrices and mathematical programming*, Springer-Verlag, Berlin, 1973.

2. KUROS, A. G., *Higher algebra*, Tankönyvkiadó, Budapest, 1968 (in Hungarian).

3. ROCKAFELLAR, R. T., *Convex analysis*, Princeton University Press, Princeton, 1970.

4. RÓZSA, P., *Linear algebra and its applications*, Tankönyvkiadó, Budapest, 1991 (in Hungarian).

5. SCHRIJVER, A., *Theory of linear and integer programming*, John Wiley & Sons, New York, 1986.

6. STRANG, G., *Linear algebra and its applications*, Academic Press, New York, 1980.

7. UJVÁRI, M., *Convex analysis*, Unpublished manuscript, 2007 (in Hungarian). URL: http://www.oplab.sztaki.hu/tanszek/letoltes.htm

8. UJVÁRI, M., A note on the graph-bisection problem, *Pure Math. and Appl.* Vol. 12 Num. 1 (2002) 119-130.

9. UJVÁRI, M., On a closedness theorem, *Pure Math. and Appl.* Vol. 15 Num. 4 (2006) 469-486.